Deterministic Algorithms for the Lovász Local Lemma[1]

Jonas Hübotter and Duri Janett
Advised by Yassir Akram

March 29, 2022

[1] David G Harris. "Deterministic algorithms for the Lovász local lemma: simpler, more general, and more parallel". In: *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM. 2022, pp. 1744–1779.

Setting

Distribution D over independent Σ-valued coordinates $X_1, \ldots, X_n$.
"Bad-events" $\mathcal{B} = \{B_1, \ldots, B_m\}$, each a boolean function of some subset of coordinates $\mathrm{var}(B_i) \subseteq \{X_1, \ldots, X_n\}$ with law p.

Example (3-SAT)

$$B_1 \doteq f_1(X_1, X_3, X_5)$$
$$B_2 \doteq f_2(X_2, X_3, X_6)$$
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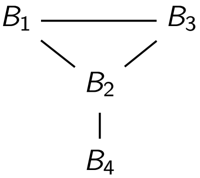

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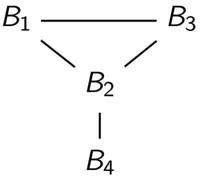

Theorem ((Symmetric) Lovász Local Lemma)

If for any i, $p(B_i) \leq p_{\max}$ and B_i affects at most d bad-events, then $ep_{\max}d \leq 1$ implies $Pr[\text{all } B_i \text{ avoided}] > 0$.

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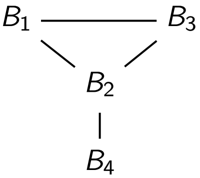

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For k-SAT and $X_i \sim \mathrm{Unif}(\{0,1\})$, $p \equiv 2^{-k}$.
$\rightsquigarrow$ satisfiable if any variable appears in at most $2^k/ke$ clauses!

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More applications:

1. Defective coloring
2. Hypergraph coloring
3. Strong coloring
4. Non-repetitive coloring
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6. Independent transversals

$\rightsquigarrow$ algorithmic versions of the Lovász Local Lemma yield automatic algorithms for these problems!

Prior Work

Algorithm: MT-Algorithm

Draw X from distribution D
while *some bad-event is true on* X **do**
  Select any true bad-event B
  For each $i \in \mathrm{var}(B)$, draw X_i from its distribution in D
end

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$\rightsquigarrow$ converges within expected polynomial time.[2]

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Prior Work

Paper	Criterion	Det.?	Parallel?
[3]	asymmetric LLL	✗	(✓)
[3]	asymmetric LLL and $d \leq \mathcal{O}(1)$	✓	(✓)
[4]	symmetric LLL with ϵ-exponential slack	✓	(✓)
[5]	Shearer criterion with ϵ-slack	✗	✓
[5]	symmetric LLL with ϵ-exponential slack and atomic bad-events	✓	✓
[6]	symmetric LLL and bad-events depend on $\mathrm{polylog}(n)$ variables	✓	✓

(✓) : under more complex conditions

[3] Robin A Moser and Gábor Tardos. "A constructive proof of the general Lovász local lemma". In: *Journal of the ACM (JACM)* 57.2 (2010), pp. 1–15.

[4] Karthekeyan Chandrasekaran, Navin Goyal, and Bernhard Haeupler. "Deterministic algorithms for the Lovász local lemma". In: *SIAM Journal on Computing* 42.6 (2013), pp. 2132–2155.

[5] Bernhard Haeupler and David G Harris. "Parallel algorithms and concentration bounds for the Lovász local lemma via witness-DAGs". In: *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM. 2017, pp. 1170–1187.

[6] David G Harris. "Deterministic parallel algorithms for fooling polylogarithmic juntas and the Lovász local lemma". In: *ACM Transactions on Algorithms (TALG)* 14.4 (2018), pp. 1–24.

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2. Faster *parallel algorithm* with simpler conditions.
3. We can ensure that the final distribution of the deterministic algorithm is not "far off" from the distribution at the end of the MT algorithm.

Plan

Alternative Characterization of MT Algorithm

Consider the resampling table R drawn according to distribution D:

	1	$\cdots$	t	$\cdots$
X_1	$*$	$*$	$*$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
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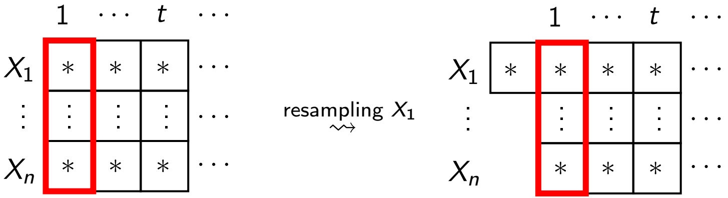

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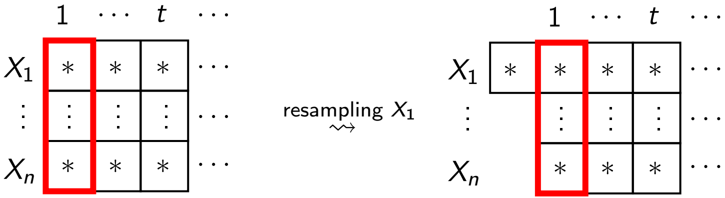

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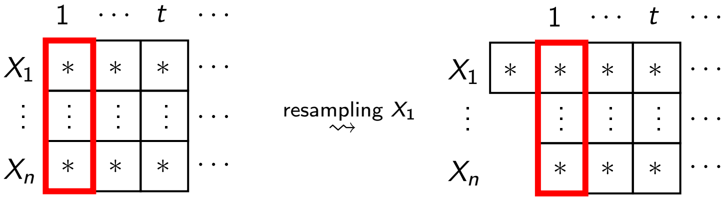

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$\rightsquigarrow$ MT algorithm deterministic with respect to resampling table!

Counting Resamples

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Why may executions be long?

Given a resampling table R, a (partial) execution of the MT algorithm is described by the sequence of resampled bad-events.

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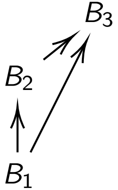

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$B_i \longrightarrow B_j$ iff $i < j$ and B_i affects B_j

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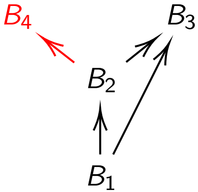

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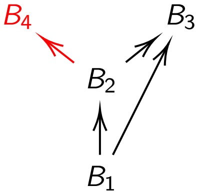

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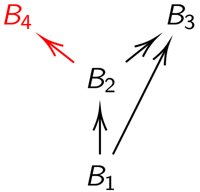

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$\rightsquigarrow$ $\hat{G}$ is always a DAG! But why are DAGs a good encoding?

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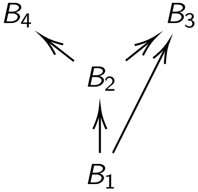

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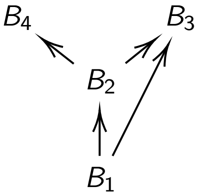

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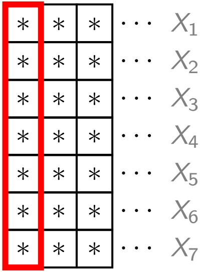

fixed resampling table R

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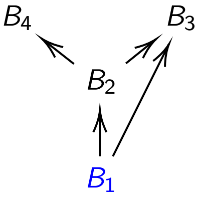

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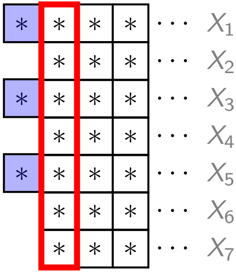

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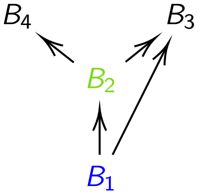

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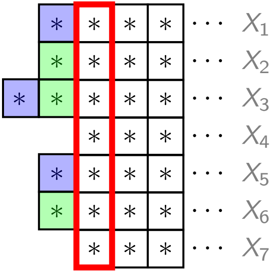

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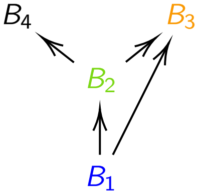

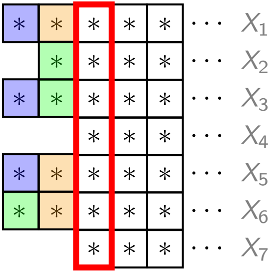

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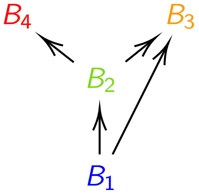

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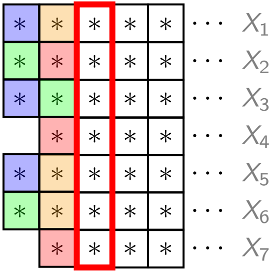

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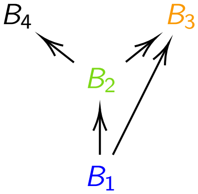

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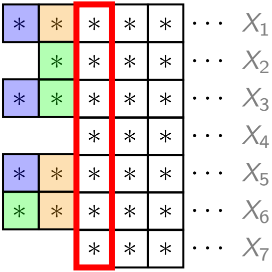

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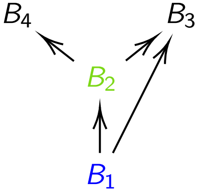

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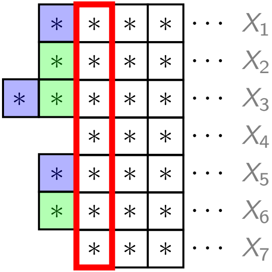

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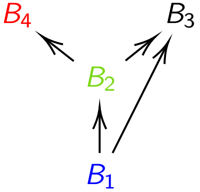

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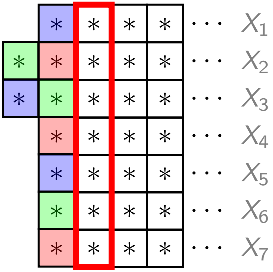

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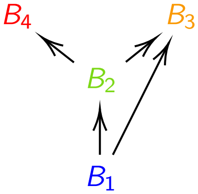

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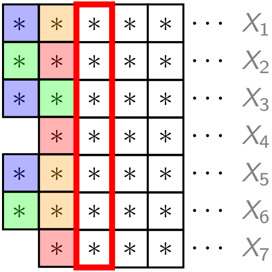

fixed resampling table R

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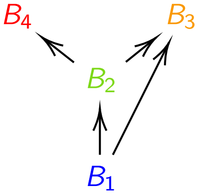
Counting Resamples

Witness DAGs encode the final configuration of the MT algorithm!



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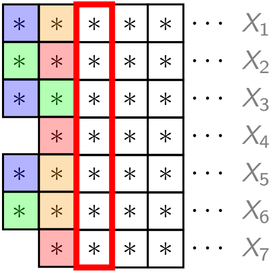

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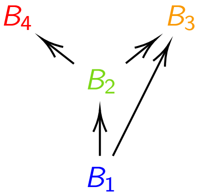

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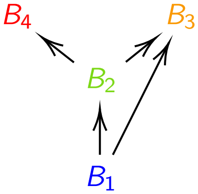

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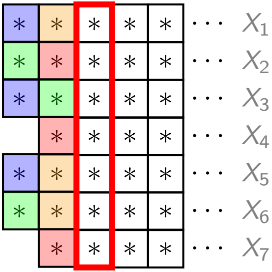

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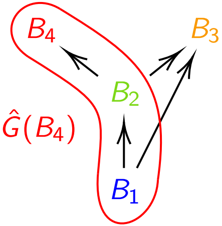

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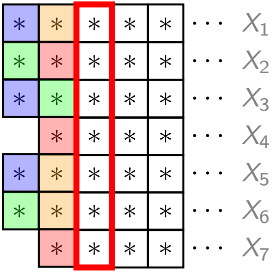

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Plan

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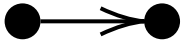

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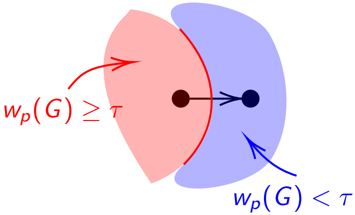

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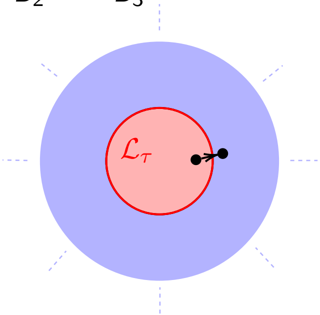

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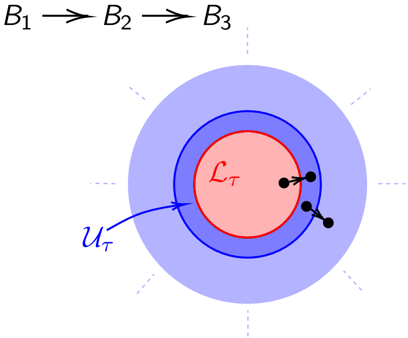

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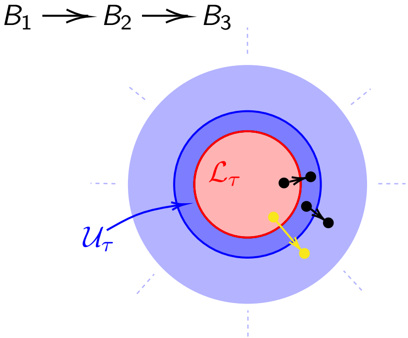

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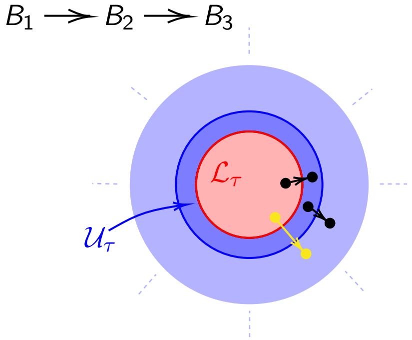

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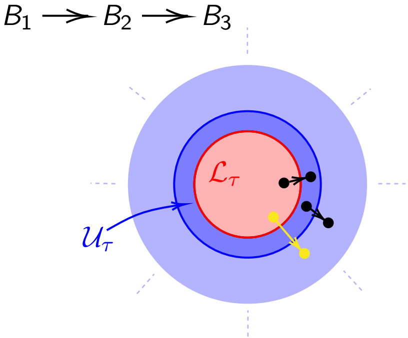

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Using the method of conditional expectation, we find R such that

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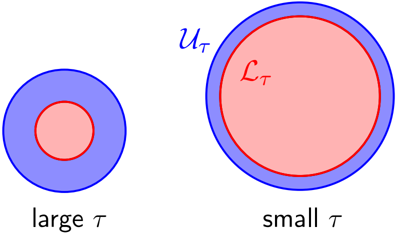

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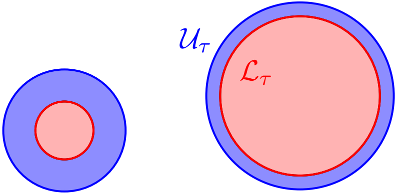

large τ | small τ

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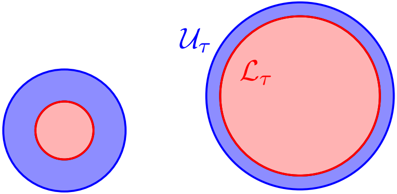

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small $\lvert\mathcal{L}_\tau\rvert$ and $\lvert\mathcal{U}_\tau\rvert$, but large $w_p(\mathcal{U}_\tau)$	large $\lvert\mathcal{L}_\tau\rvert$ and $\lvert\mathcal{U}_\tau\rvert$, but small $w_p(\mathcal{U}_\tau)$ if $w_p(\mathcal{C}) < \infty$

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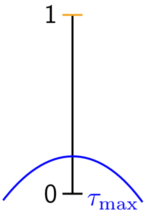
Use exponential backoff!
Example $\tau = 2^0 = 1$.



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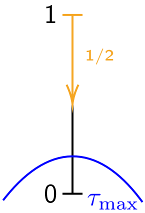

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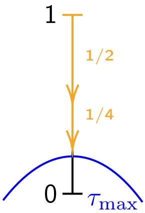

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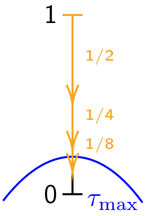

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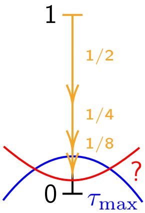
Are $\mathcal{U}_\tau$ and $\mathcal{L}_\tau$ of polynomial size?



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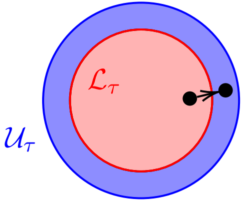

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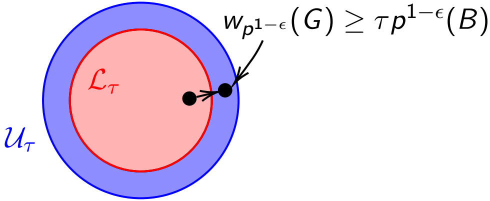

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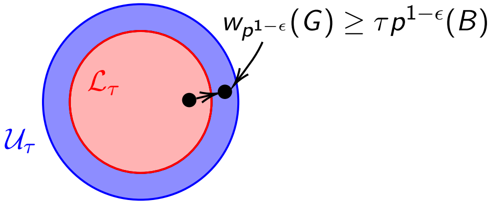

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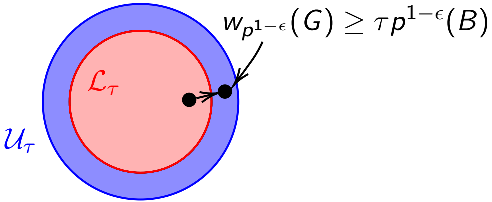

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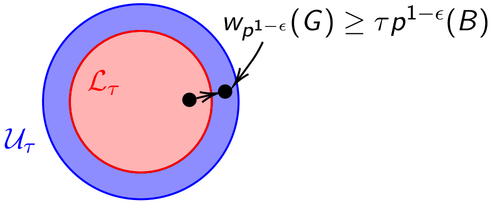

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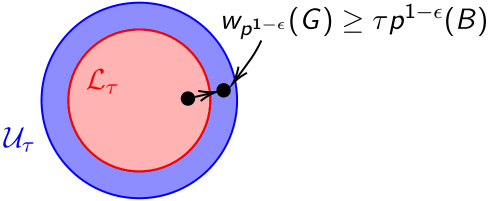

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W_ϵ is polynomial under common LLL conditions!

The Algorithm

Algorithm: Deterministic MT-Algorithm

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Using exponential backoff, select "large" τ such that $w_p(\mathcal{U}_\tau) < 1$
Using method of conditional expectations, find resampling table R avoiding $\mathcal{U}_\tau$
Run the deterministic MT algorithm on R

We have seen that the final step takes at most $|\mathcal{G}[R]| \leq |\mathcal{L}_\tau[R]|$ iterations!

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This algorithm does not cover some scenarios:

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Thanks for your attention! Questions?

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Also need to generate $\mathcal{U}_\tau$, which can be done in $\mathrm{poly}(|\mathcal{U}_\tau|)$ time.

Polynomial Bound of ${w_{p^{1-\epsilon}}(\mathcal{G}_B)}/{p^{1-\epsilon}(B)}$

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$$\begin{aligned}\mu^{(h+1)}(B) &= p(B) \sum_{J \in \mathsf{Stab}(B)} \mu^{(h)}(J) \\ &\leq p(B) \sum_{J \subseteq \bar{\Gamma}(B)} \prod_{B' \in J} \mu^{(h)}(B')\end{aligned}$$

$$\begin{aligned}\sum_{J \subseteq \bar{\Gamma}(B)} \prod_{B' \in J} ep(B') &\leq \sum_{J \subseteq \bar{\Gamma}(B)} (ep_{\max})^{|J|} \leq \sum_{k=0}^{d} \binom{d}{k} (ep_{\max})^k \\ &= (1 + ep_{\max})^d \leq \exp(\underbrace{ep_{\max} d}_{\leq 1}) \leq e.\end{aligned}$$