Probabilistic Social Choice
(Brandl, Brandt, and Seedig 2016)

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Ava, Ben, and Chloe want to go for lunch.

Ava wants to go to Klara's Kitchen $\succ$ Mensa $\succ$ foodLAB.
Ben wants to go to Mensa $\succ$ foodLAB $\succ$ Klara's Kitchen.
Chloe wants to go to foodLAB $\succ$ Klara's Kitchen $\succ$ Mensa.

So where to go?

Aggregating the preferences of individuals
is the problem of social choice.

Framework of Social Choice

Given

- an electorate (Ava, Ben, and Chloe),
- a set of alternatives $A = \{\text{foodLAB}, \text{Klara's Kitchen}, \text{Mensa}\}$,
- a preference profile R over A

$1/3$	$1/3$	$1/3$
K	M	F
M	F	K
F	K	M

find a social choice function $f : \mathcal{R}_A{}^1 \to A$ that returns a single alternative,
i.e., identifies a place for lunch that is in A.

[1] $\mathcal{R}_A = \Delta(\mathcal{L}(A))$ is the set of all preference profiles over A. Here, $\mathcal{L}(A)$ is the set of all linear preference relations over A and $\Delta(S)$ is the set of all probability distributions over S.

Framework of Social Choice

$R(a \succ b)$ is the fraction of the electorate that prefers a to b.

Example

$1/3$	$1/3$	$1/3$
K	M	F
M	F	K
F	K	M

$$R(K \succ M) = 2/3$$
$$R(M \succ F) = 2/3$$
$$R(F \succ K) = 2/3$$

Framework of Social Choice

We do not just want *some* social choice function,
we want a *good* social choice function.

We need an *axiomatic characterization*.

Condorcet consistency

If $R(a \succ b) > 1/2$ for any $b \in A \setminus \{a\}$, then a is called the Condorcet winner.
f is Condorcet consistent if it always chooses the Condorcet winner provided one exists.

Example

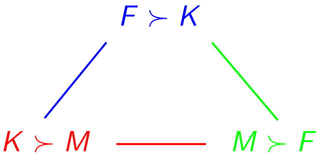
$2/3$	$1/3$
K	F
M	K
F	M

$$R(K \succ M) = 1$$
$$R(K \succ F) = 2/3$$

$\rightsquigarrow$ K is the Condorcet winner

Framework of Social Choice

$1/3$	$1/3$	$1/3$
K	M	F
M	F	K
F	K	M



The preferences of Ava, Ben, and Chloe do not yield a Condorcet winner (known as the Condorcet paradox).

Many commonly used social choice functions are Condorcet inconsistent.
(plurality, two-round runoff, instant runoff, Borda count)

$\rightsquigarrow$ deterministic social choice functions are problematic

Probabilistic Social Choice

Instead of returning a single alternative, a probabilistic social choice function returns probability distributions (lotteries) over alternatives.

Formally, $f : \mathcal{R}_A \to \mathcal{P}(\Delta(A))$ where $\Delta(A)$ is the set of lotteries over A.

Example (random dictatorship)

Choose a preference relation among all preference relations uniformly at random, and choose its most-preferred alternative.

$1/3$	$1/3$	$1/3$
K	M	F
M	F	K
F	K	M

$\rightsquigarrow RD(R) = \{1/3 F + 1/3 K + 1/3 M\}$

Probabilistic Social Choice

Requirements

- when Ava goes to lunch on her own, she should choose Klara's Kitchen with probability 1 (unanimity)
- when f returns multiple lotteries for a preference profile, there is an "arbitrarily close" preference profile that only yields a single lottery (decisiveness)

$\rightsquigarrow$ deterministic social choice functions are a special case of probabilistic social choice functions which only return *degenerate* lotteries (i.e., lotteries that put all probability mass on a single alternative).

Consistency Axioms

1/2	1/2
K	M
M	F
F	K

R'

1/2	1/2
K	M
F	F
M	K

R''

1/4	1/4	1/2
K	K	M
M	F	F
F	M	K

$1/2R' + 1/2R'' = R$

When $1/2K + 1/2M$ is chosen by R' and R'', then it must also be chosen by R.

Population Consistency

When two separate electorates R' and R'' agree on a lottery, this lottery must also be chosen by a combination of the electorates.
$\rightsquigarrow$ consistency with respect to a variable electorate

Consistency Axioms

Ben *really* wants to go to Mensa so he devises an evil plan to confuse his friends.
Instead of proposing to go to Mensa, he introduces separate clones $B = \{M1, M2, \dots\}$ for table 1 at Mensa, table 2 at Mensa, ...

$1/3$	$1/3$	$1/3$
K	M1	F
M1	M2	K
M2	F	M2
F	K	M1

R

$1/3$	$1/3$	$1/3$
K	M1	F
M1	F	K
F	K	M1

$R|_{A'}$
$A' = \{F, M1, K\}$

$2/3$	$1/3$
M1	M2
M2	M1

$R|_B$
$B = \{M1, M2\}$

Cloning Consistency

We should assign the same probability to M independently of the number of clones and the internal relationship between these clones.
$\rightsquigarrow$ consistency with respect to components of *similar* alternatives

Consistency Axioms

$1/3$	$1/3$	$1/3$
K	M1	F
M1	M2	K
M2	F	M2
F	K	M1

R

$1/3$	$1/3$	$1/3$
K	M1	F
M1	F	K
F	K	M1

$R|_{A'}$
$A' = \{F, M1, K\}$

$2/3$	$1/3$
M1	M2
M2	M1

$R|_B$
$B = \{M1, M2\}$

f should assign more probability to $M1$ than to $M2$.

Composition Consistency (extending cloning consistency)

The probability of clones must be directly proportional to the probabilities that f assigns to these alternatives for $R|_B$.

We have seen that *Condorcet consistency*, *population consistency*, and *composition consistency* arise as natural axioms.

Deterministic social choice functions cannot satisfy

- Condorcet consistency and population consistency
- population consistency and composition consistency

What about probabilistic social choice functions?

- random dictatorships satisfy population consistency and cloning consistency
- *maximal lotteries* satisfy population consistency and composition consistency (and Condorcet consistency) (Brandl, Brandt, and Seedig 2016)

$\rightsquigarrow$ maximal lotteries settle a dispute between the founding fathers of social choice theory, Chevalier de Borda and Marquis de Condorcet.

Random aggregation does not entail decisions must lack rationale, but rather that results may be random in the absence of a clear winner and as such the procedure is harder to exploit.

"Sometimes, in fact, we want to exclude reasons from decision-making entirely. This is what lotteries can do."
(Stone 2011)

Acceptability of Lotteries

Amidst which circumstances are random choices useful?

- depending on the risk aversion of voters, and
- the effective degree of randomness.

Acceptability of Lotteries

Risk Aversion

Typically, the risk aversion depends on

- the stakes of the decision, and
- the frequency of preference aggregation.

If preferences are aggregated rarely the law of large numbers does not apply and a risk-averse voter might prefer a sure outcome to a random outcome even if in expectation she prefers the random outcome.

In the Athenian democracy, lotteries were a principal component.[2] The process of selecting candidates by a lottery for a political office is also known as *sortition* or *selection by lot*.

[2] Special allotment machines (*kleroteria*) were used to select candidates.

Acceptability of Lotteries

Effective Degree of Randomness

What to do about ties? Are ties rare or common?

Condorcet view: we encounter a tie iff there is no Condorcet winner, i.e., no unequivocal winner.

It can be shown that Condorcet's view of equivocality and the randomness of maximal lotteries coincide.[3]

Empirically, there often exists a Condorcet winner. If there exists no Condorcet winner, the support of maximal lotteries is typically small compared to the number of alternatives.

[3] Maximal lotteries are degenerate iff there is a Condercet winner.

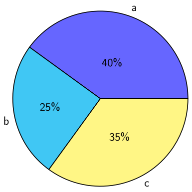
An Alternative Interpretation

We have been thinking about lotteries as probability distributions over alternatives, used to obtain a single (or multiple) choices.

Example

Consider a firm that asks its employees to spend a fixed budget on non-profits of their choice. How can we come up with a fair allocation of the budget?

$0.4a + 0.25b + 0.35c \quad \rightsquigarrow$



Probabilities can equivalently be interpreted as fractions of a divisible resource such as money or time.

Thank you for your attention

How often do preferences have to be (repeatedly) aggregated until lotteries become acceptable?

Is randomness in decision making at all acceptable?
Is it desirable?

References

Zeckhauser, Richard (1969). "Majority rule with lotteries on alternatives". In: *The Quarterly Journal of Economics*, pp. 696–703.

Stone, Peter (2011). *The luck of the draw: The role of lotteries in decision making*. Oxford University Press.

Brandl, Florian, Felix Brandt, and Hans Georg Seedig (2016). "Consistent probabilistic social choice". In: *Econometrica* 84.5, pp. 1839–1880.

The Inefficiency of Majority Rule without Lotteries

The 101 Club must choose a single form of entertainment for all club members. The membership rolls contain fifty football fanatics, fifty ballet aficionados, and a single lover of musical comedy. For the footballers the musical is almost as bad as the ballet. For the ballet enthusiasts the musical is little better than football. By majority rule, using pairwise comparisons without lotteries, the musical will be chosen. If lotteries were permitted, a fifty-fifty football-ballet lottery would defeat the musical by any required plurality up to 100 out of 101. (Zeckhauser 1969)

The Inefficiency of Majority Rule without Lotteries

> *A political system, such as the one in the United States, which rules out lotteries might lead to dominance by the center, even if the right and left could put together a majority coalition that would prefer a lottery on the extremes to a middle outcome. (Zeckhauser 1969)*

In general, majority rule without lotteries corrects towards the median choice.
This can be considered inefficient or desirable. The judgement may depend on the setting.

Maximal Lotteries

For a set of alternatives A and preference profile $R \in \mathcal{R}_A$, let M_R be the majority margin matrix defined by

$$M_R(x, y) = R(x \succ y) - R(y \succ x).$$

$M_R(x, y)$ denotes the difference between the fraction of voters preferring x to y and the fraction of voters preferring y to x.
$\rightsquigarrow$ M_R is skew-symmetric

Any $x \in A$ such that $M_R(x, y) \geq 0$ for all $y \in A$ is a *(weak) Condorcet winner*.
Any $x \in A$ such that $M_R(x, y) > 0$ for all $y \in A \setminus \{x\}$ is a *strict Condorcet winner*.

Maximal Lotteries

We fix two lotteries $p, q \in \Delta(A)$.
For p and q, the majority margin M_R can be extended to the expected majority margin $p^T M_R q$.
The set of maximal lotteries is then the set of *probabilistic Condorcet winners*:

$$ML(R) = \{p \in \Delta(A) \mid \forall q \in \Delta(A).\ p^T M_R q \geq 0\}.$$

Interpretation in the Context of Game Theory

Let M_R be the payoff matrix of a symmetric zero-sum game.
Lotteries in social choice theory correspond to mixed strategies in game theory.
Then, maximal lotteries correspond to the mixed Nash equilibria (i.e., maximin strategies).
$\rightsquigarrow$ efficient computation using linear programming.

Maximal Lotteries

Example

$1/2$	$1/3$	$1/6$
a	a	b
b	c	c
c	b	a

R

$$M_R = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1/3 & 1/3 \\ -1/3 & 0 & 1/3 \\ -1/3 & -1/3 & 0 \end{pmatrix} \end{array}$$

Clearly, a is the only Nash equilibrium of the symmetric zero-sum game, so $ML(R) = \{a\}$ (a is also the Condorcet winner).
Note that a random dictatorship returns the lottery $5/6a + 1/6b$.

Randomness of Maximal Lotteries for Two Alternatives

Let $A = \{x, y\}$ and $R \in \mathcal{R}_A$.

$$ML(R) = \begin{cases} \{x\} & \text{if } R(x \succ y) > 1/2 \\ \{y\} & \text{if } R(x \succ y) < 1/2 \\ \Delta(A) & \text{otherwise} \end{cases}$$

In contrast, a random dictatorship returns the lottery $R(x \succ y)x + R(y \succ x)y$.

$\rightsquigarrow$ in a way, maximal lotteries have the smallest reasonable degree of randomness while random dictatorships are the most random.