Theoretical Computer Science
Complexity Theory

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Outline

$\mathcal{P}$

$\mathcal{NP}$

- Polynomially Bounded Verifier
- Polynomial Reduction
- $\mathcal{NP}$-completeness

Approximation Algorithm

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where $\text{TIME}(\mathcal{O}(f)) = \bigcup_{g \in \mathcal{O}(f)} \text{TIME}(g)$.

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A problem A is in $\mathcal{NP}$ if and only if solutions (which are described by *certificates*) to the problem can be verified in polynomial time by a DTM (a *polynomially bounded verifier*).

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- However, once the NTM found a path, the length of this path must be polynomial in the size of the input.
- Thus, a DTM must be able to verify that a given path is correct in polynomial time.

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- M is a polynomially bounded verifier for the language $\{w \in \Sigma^* \mid \exists c \in \Delta^*.\ w\#c \in L(M)\}$ (i.e. the language of all words that have a certificate) if there exists a polynomial p such that $\text{time}_M(w\#c) \leq p(|w|)$.

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Especially: $c \leq p(|w|)$, i.e. the size of the certificate must be polynomially bounded by the size of the input.

Polynomial Reduction

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Given problems $A \subseteq \Sigma^*, B \subseteq \Gamma^*$, A is polynomially reducable to B (denoted $A \leq_p B$) if there exists a total and by a DTM in polynomial time computable function $f : \Sigma^* \rightarrow \Gamma^*$ such that

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The complexity classes $\mathcal{P}$ and $\mathcal{NP}$ are closed under polynomial reduction.

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- If $A \leq_p B$ and $B \in \mathcal{NP}$, then $A \in \mathcal{NP}$.
- If there exists a polynomially bounded verifier for A, then $A \in \mathcal{NP}$.

Approximation Algorithm

Definition 8

A *d*-approximation algorithm ($d \in \mathbb{R}$) for an optimization problem is an algorithm that computes in polynomial time a solution to the problem that is at most d times worse than the optimal solution.