Theoretical Computer Science
Regular Languages

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- Interlude: Structural Induction
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Overview

Representations of regular languages

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- ϵ-NFA
- Regular Expression (Regex)

DFA

Definition 1

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- a (finite) alphabet Σ;
- a total transition function $\delta : Q \times \Sigma \to Q$;
- an initial state $q_0 \in Q$; and
- a set of terminal (accepting) states $F \subseteq Q$.

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The language accepted by M is $L(M) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\}$.

NFA

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- Q, Σ, q_0, F as defined for DFAs; and
- a (partial) transition function $\delta : Q \times \Sigma \to 2^Q$.

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The language accepted by N is
$L(N) = \{w \in \Sigma^* \mid \hat{\bar{\delta}}(\{q_0\}, w) \cap F \neq \emptyset\}$.

NFA $\rightarrow$ DFA (determinization)

Idea

Interpret every reachable subset $S \subseteq 2^Q$ in the NFA N as its own state in the new DFA M.

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Worst-case exponential growth!

ϵ-NFA

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An ϵ-NFA $N = (Q, \Sigma, \delta, q_0, F)$ is an NFA with a special symbol $\epsilon \neg \in \Sigma$ where

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ϵ-transitions can be executed at any time without reading a symbol.

ϵ-NFA $\to$ NFA

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Given ϵ-NFA $N = (Q, \Sigma, \delta, q_0, F)$ construct NFA $N' = (Q, \Sigma, \delta', q_0, F')$ where

$$\delta' : Q \times \Sigma \rightarrow 2^Q : (q, a) \mapsto \bigcup_{i,j \geq 0} \hat{\delta}(\{q\}, \epsilon^i a \epsilon^j);$$

if $\epsilon \in L(N)$ then $F' = F \cup \{q_0\}$ else $F' = F$.

Product-Construction

Idea

Given DFAs $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

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For the product automaton $L(M) = L(M_1) \cap L(M_2)$ holds.

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Algorithm ($\mathcal{O}(|Q|^2)$ for constant $|\Sigma|$)

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3. merge equivalent states

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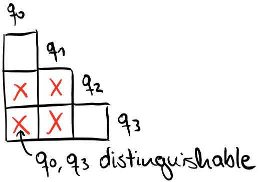

1. mark all pairs $p, q \in Q$ if $p \in F$ and $q \in Q \setminus F$
2. while $\exists$ unmarked $\{p, q\}$ and $\exists a \in \Sigma$, if $\{\delta(p, a), \delta(q, a)\}$ is marked, mark $\{p, q\}$

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$[a]_\sim = \{b \mid a \sim b\}$ is called the equivalence class of a under $\sim$.

The set of equivalence classes $A/{\sim} = \{[a]_\sim \mid a \in A\}$ is called the quotient set of $\sim$.

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with $\delta'([p]_{\equiv_M}, a) = [\delta(p, a)]_{\equiv_M}$ for $p \in Q, a \in \Sigma$.

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Definition 9

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It follows that $\hat{\delta}([\epsilon]_{\equiv_L}, w) = [w]_{\equiv_L}$ for $w \in \Sigma^*$, hence $L(M_L) = L$.

Theorem of Mihill-Nerode

Theorem 10 (Theorem of Mihill-Nerode)

$L \subseteq \Sigma^*$ is regular $\iff$ $\equiv_L$ has finitely many equivalence classes.

DFA $\rightarrow$ RLG

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Then, $L(G) = L(M)$.

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- $Y \in \delta(X, a)$ iff $(X \to aY) \in P$

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  - α^* (repetition).

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- $L(\alpha|\beta) = L(\alpha) \cup L(\beta)$; and
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- $P(\alpha_i)$ for $i \in \{1, \ldots, k\}$; and
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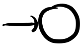

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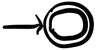
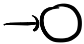
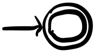


Regex → ϵ-NFA (Kleene)

$\emptyset$



ϵ



a

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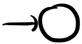
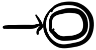
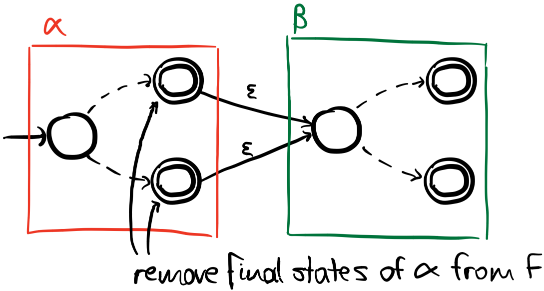


ϵ



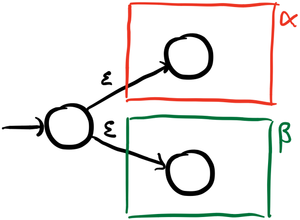
a

$\alpha\beta$



Regex $\to$ ϵ-NFA (Kleene)

$\alpha \mid \beta$



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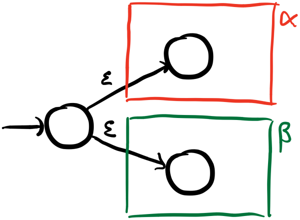
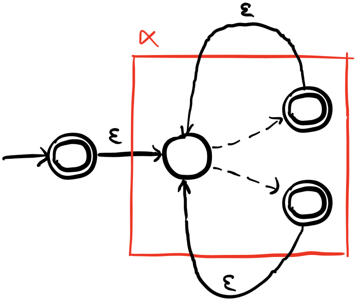


α^*



DFA/NFA $\to$ Regex (Kleene)

Given $M = (Q, \Sigma, \delta, q_1, F)$ with $Q = \{q_1, \ldots, q_n\}$ define

$$R_{ij}^k = \{w \in \Sigma^* \mid \text{input } w \text{ transitions from } q_i \text{ to } q_j \text{ and all states in between have an index } \leq k\}.$$

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Idea: for all $i, j \in [n]$ and $k \in [n]_0$
a regex α_{ij}^k can be constructed with $L(\alpha_{ij}^k) = R_{ij}^k$.

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Induction over k.

- $k = 0$:

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$$R_{ij}^{k+1} = R_{ij}^k \cup R_{i(k+1)}^k (R_{(k+1)(k+1)}^k)^* R_{(k+1)j}^k$$

New paths using q_{k+1} in terms of the already built subpaths.

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We now have, $L(M) = L(\alpha_{1i_1}^n \mid \cdots \mid \alpha_{1i_r}^n)$ where $\{q_{i_1}, \ldots, q_{i_r}\} = F$.

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Theorem 11 (Arden's Lemma for regular languages)

Let A, B, X be regular languages and $\epsilon \notin A$, then:

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Theorem 12 (Arden's Lemma for regular expressions)

Let α, β, X be regular expressions and $\epsilon \notin L(\alpha)$, then:

$$X = \alpha X \mid \beta \implies X = \alpha^*\beta.$$

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Given the regular languages R, R_1, R_2, then the following are also regular languages:

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- $R_1 \setminus R_2$.

Pumping Lemma

Lemma 14 (Pumping Lemma for regular languages)

Let $R \subseteq \Sigma^*$ *be regular.*

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A necessary condition for regular languages.

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Then, $\forall i \geq 0.\ uv^i w \in L$.
Now, use the last statement to find a contradiction.